J. Fluid Mech. (1973), *vol.* 61, *part* 1, *pp.* 65–72
Printed in Great Britain

Plane jet flows from slowly varying nozzles

By M. B. LESSER
Institut CERAC S.A., Ecublens, Switzerland

(Received 17 September 1972 and in revised form 21 June 1973)

The solution to the steady problem of an inviscid jet emerging from a symmetric nozzle of slowly varying profile is sought as an asymptotic series in the wall slope. The expansion of the solution in the region near the nozzle lip is singular at infinity, so that a matched expansion technique is evolved to solve the problem. To the order to which the solution is obtained in the present paper, the jet contraction ratio is shown to be the same as that from a nozzle formed by two inclined planes with inclination angle the same as the exit slope of the nozzle. Composite expansions are formed and used to check the consistency of expansion and matching procedures.

1. Introduction

The calculation of the plane jet flows of an incompressible fluid by conformal mapping methods has become an established technique of fluid mechanics. Results of such calculations, however, are seldom available in explicit or readily useful form. Despite the existence of the 'exact' theories, there appears to still be a need for efficient approximate methods, even in plane jet flow calculations. The purpose of the present paper is to show how the currently popular method of matched asymptotic expansions can be applied to this class of problems, in particular how the matching principle of Van Dyke (1964) can be used to find a uniformly valid asymptotic solution for the jet emission from a varying nozzle having a small wall slope. The contraction coefficient will be calculated to $O(\epsilon)$ (ϵ = measure of wall slope), and hence we shall see that to this order the contraction is the same as that from a nozzle formed by two inclined planes. Once we have the machinery of the matched expansion method it is possible to construct composite expansions, and in the relatively simple problem considered the consistency check advocated by Fraenkel (1969) can be employed to show that the composite expression uniformly satisfies all the conditions of the problem.

2. Formulation and outer expansions

For our demonstration of how matched expansion methods apply to jet flows we consider the configuration depicted in figure 1. Inviscid incompressible flow with speed U_0' enters the nozzle at $X = -\infty$ (the nozzle shape is asymptotic to a parallel section of half-width H as $X \to -\infty$). The symmetric nozzle shape varies over a length scale L, hence the nozzle profile has the form $Y = Hh(X/L)$. The nozzle slope is $(H/L)\, h'(X/L)$, and h' is assumed to be $O(1)$. The shape

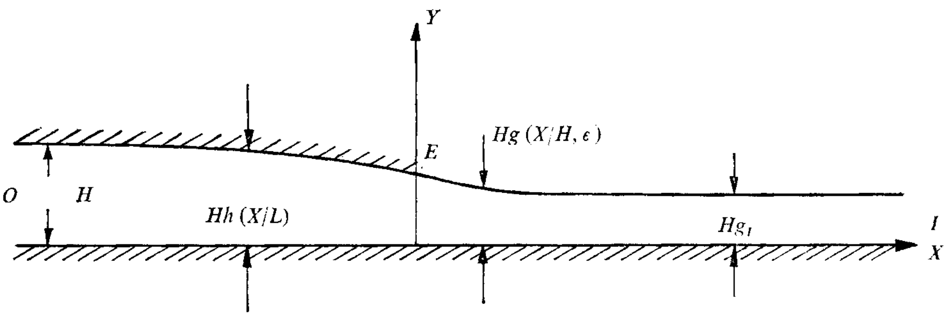

FIGURE 1. Strip-like flow domain.

parameter $\epsilon = H/L$ is chosen as the perturbation expansion parameter. The nozzle width is assumed to be monotonically decreasing or if it is increasing the rate of increase is such that the flow does not separate from the nozzle walls. The jet exits at point E and contracts as $X \to +\infty$ (point I), where the half-width attains its asymptotic value $g_I(\epsilon)\,H$, i.e. the final half-width depends on the parameter ϵ, g_I being an unknown function of ϵ. For convenience we represent the nozzle shape and jet free surface by the function

$$Y = H\tilde{h}(X, \epsilon),$$

where $$\tilde{h}(X \leqslant 0) = h(X/L)$$

and $$\tilde{h}(X \geqslant 0) = g(X/H, \epsilon).$$

The function h (nozzle shape) is given, but the free surface g must be found as part of the solution procedure. The functional form assumed for g, i.e. use of the variable X/H, is chosen because the free-surface shape is expected to vary on the length scale of the nozzle width.

We first formulate the problem in non-dimensional 'outer' variables appropriate to the region bounded away from E. The outer variables will be seen to lead to the so-called hydraulic approximation appropriate to the 'slowly varying' nozzle profile. The requirement that the boundary curve $\tilde{h}$ be a streamline leads to the conclusion that $V/U = O(\epsilon)$ (V being the velocity in the Y direction). Thus, if velocities are scaled with the entrance velocity U_0, the outer variables are

$$\bar{x} = X/L, \quad \bar{y} = Y/H,$$

$$\bar{Z} = \bar{x} + i\epsilon\bar{y},$$

$$\bar{u} = U/U_0, \quad \bar{v} = V/(\epsilon U_0),$$

$$\bar{\phi} = \Phi/(U_0 L), \quad \bar{\psi} = \Psi/(U_0 L)$$

and $$\bar{F}(\bar{z}) = \bar{\phi} + i\bar{\psi},$$

where $\bar{\phi}$ is the outer velocity potential, $\bar{\psi}$ the outer stream function and $\bar{F}(\bar{z})$ is the complex velocity potential. From these definitions

$$\bar{F}'(\bar{z}) = \bar{q} = \bar{u} - i\epsilon\bar{v},$$

where $\bar{q}$ is the complex velocity.

The problem is to find the free boundary $g(\bar{x}/\epsilon, \epsilon)$ and the complex velocity $\bar{q}(\bar{x};\, \epsilon)$, the latter being analytic in the strip

$$\{\bar{z}\colon -\infty < \bar{x} < +\infty, 0 < \bar{y} < \tilde{h}(\bar{x}, \epsilon)\}$$

and satisfying the conditions

$$\bar{q}(\bar{z}; \epsilon) = [1 - i\epsilon h'(\bar{x})]\,\bar{u} \quad \text{on} \quad \bar{z} = \bar{x} + i\epsilon h(\bar{x}), \quad \bar{x} < 0, \tag{2.1}$$

$$\left.\begin{aligned} \bar{q}(\bar{z};\, \epsilon) &= [1 - i\epsilon\, dg/dx]\,\bar{u} \\ |\bar{q}|^2 &= |u_I(\epsilon)|^2 = |g_I(\epsilon)|^{-2} \end{aligned}\right\} \quad \text{on} \quad \bar{z} = \bar{x} + i\epsilon g(\bar{x}/\epsilon, \epsilon), \qquad (2.2a), (2.2b)$$

also
$$\bar{q} = \bar{v} = 0 \quad \text{on} \quad \bar{z} = \bar{x}, \tag{2.3}$$

$$\bar{q} \to 1 \quad \text{as} \quad \bar{x} \to -\infty \quad (\text{the point } O), \tag{2.4}$$

$$\bar{q} \to u_I(\epsilon) \quad \text{as} \quad \bar{x} \to +\infty \quad (\text{the point } I) \tag{2.5}$$

and
$$\bar{q} = O(1) \quad \text{as} \quad \bar{z} \to i\epsilon h(0) = i\epsilon h_E \quad (\text{the point } E). \tag{2.6}$$

Two conditions $(2.2a, b)$ are required on the free surface because g must be determined. The final jet velocity and jet width $u_I(\epsilon)$ and $g_I(\epsilon)$ are also unknown but related by the continuity condition. Also we tentatively assume (later calculation will show this to hold) that

$$\lim_{\epsilon \to 0,\, \bar{x} > 0} g(\bar{x}/\epsilon, \epsilon) = g_I(\epsilon) + O(e^{-\bar{x}\alpha/\epsilon}), \tag{2.7}$$

where $\alpha > 0$ is a constant. The outer limit is defined as

$$\lim_{\bar{z}\ \text{fixed},\, |\bar{x}| > 0} \bar{q}(\bar{z};\, \epsilon) = \bar{q}^{(0)}(\bar{z})$$

and for notational convenience we shall adopt the outer partial expansion operator (see Fraenkel 1969)

$$\bar{E}_1 \bar{q}(\bar{z};\, \epsilon) = \bar{q}^{(0)} + \epsilon \bar{q}^{(1)}.$$

In the present problem the expansion is only carried out to $O(\epsilon)$ and it is clear, at least to this order, that the power-series form in ϵ is the choice that permits matching. The complex velocity $\bar{q}$ is thus

$$\bar{E}_1 \bar{q} = \bar{q}^{(0)} + \epsilon \bar{q}^{(1)},$$

plus terms $O(\epsilon^2)$. Inserting $\bar{z} = \bar{x} + i\epsilon\bar{y}$ and expanding, we find

$$\bar{q}(\bar{x} + i\epsilon\bar{y}) = \bar{q}^{(0)}(\bar{x}) + \epsilon[\bar{q}^{(1)}(\bar{x}) + i\bar{q}^{(0)\prime}(\bar{x})\,\bar{y}] + \epsilon^2[\bar{q}^{2)}(\bar{x}) - \tfrac{1}{2}\bar{q}^{(0)\prime\prime}(\bar{x})\,\bar{y}^2 + i\bar{y}\bar{q}^{(1)\prime}(\bar{x})] + O(\epsilon^3).$$

Expanding the boundary condition on $\bar{y} = \tilde{h}(\bar{x}, \epsilon)$ and equating terms shows that

$$(h\bar{q}^{(0)})' = (h\bar{q}^{(1)})' = 0, \quad \bar{x} < 0,$$

and
$$(g_I^{(0)}\bar{q}^{(0)})' = (g_I^{(1)}\bar{q}^{(1)})' = 0, \quad \bar{x} > 0,$$

where
$$g_I(\epsilon) = g_I^{(0)} + \epsilon g_I^{(1)} + \dots.$$

At this stage in the calculation the $g_I^{(n)}$ *are not known.* The boundary condition at O, that $\bar{u} \to 1$, however, allows one completely to determine $\bar{q}^{(0)}$ and $\bar{q}^{(1)}$ for $\bar{x} < 0$, thus

$$\bar{q}(\bar{z}) = 1/h(\bar{z}) + O(\epsilon^2), \quad \bar{x} < 0, \tag{2.8}$$

and

$$\bar{q}(\bar{z}) = \frac{1}{g_I^{(0)}} - \epsilon\frac{g_I^{(1)}}{g_I^{(0)2}} + O(\epsilon^2). \tag{2.9}$$

The outer expansion for $\bar{q}(\bar{z})$ $(x > 0)$ was determined by using the fact that

$$\bar{u}_I = (g_I^{(0)} + \epsilon g_I^{(1)})^{-1} + O(\epsilon^2).$$

The expansions for $\bar{x} < 0$ and $\bar{x} > 0$ can not be matched, hence the constants $g_I^{(0)}$ and $g_I^{(1)}$ must be determined by matching with an inner expansion.

3. Inner region and matching

The length scale in the inner region is H and in the nozzle exit region *there is no reason to expect* $V/U = O(\epsilon)$. Therefore, we define the inner variables as

$$x = \bar{x}/\epsilon, \quad y = \bar{y}, \quad z = \bar{z}/\epsilon = x + iy,$$

$$u = \bar{u}, \quad v = \epsilon\bar{v}, \quad \phi = \bar{\phi}, \quad \psi = \bar{\psi},$$

$$F = \phi + i\psi = \bar{F}, \quad F' = \epsilon q = \epsilon(u - iv).$$

The inner partial expansion operator is defined as $E_1 q = q^{(0)} + \epsilon q^{(1)}$, and

$$q = q^{(0)} + \epsilon q^{(1)} + O(\epsilon^2).$$

The conditions for the inner problem on the boundary of the strip domain

$$\{z\colon -\infty < x < +\infty,\ 0 < y < \tilde{h}(\epsilon x;\ \epsilon)\}$$

are

$$q = [1 - i\epsilon h'(\epsilon x)]\,u \quad \text{on} \quad z = x + ih(\epsilon x), \quad x < 0, \tag{3.1}$$

$$\left.\begin{aligned} q &= [1 - i\,dg/dx]\,u \quad &(3.2a)\\ |q|^2 &= u_I^2(\epsilon), \quad &(3.2b)\end{aligned}\right\} \quad \text{on} \quad z = x + ig(x, \epsilon), \quad x > 0,$$

$$v = 0 \quad \text{on} \quad z = x \tag{3.3}$$

and

$$|q|^2 = O(1) \quad \text{as} \quad z \to ih_E. \tag{3.4}$$

Conditions for $|x| \to \infty$ are found from the requirement that the inner expansion match with the outer along the real axis. Straightforward substitution of the expansion $E_1 q$ into the above conditions leads to the result that

$$\bar{q}^{(0)} = \text{constant}$$

and $\bar{q}^{(1)}$ must satisfy the requirements

$$v^{(1)} = h'_E/h_E, \quad \text{where} \quad h'_E = h'(0), \quad \text{on} \quad z = x + ih_E, \quad x < 0, \tag{3.5}$$

$$\left.\begin{aligned} v^{(1)} &= g^{(1)\prime}(x)/h_E \quad &(3.6a)\\ u^{(1)} &= u_I^{(1)} \quad &(3.6b)\end{aligned}\right\} \quad \text{on} \quad z = x + ih_E, \quad x > 0,$$

$$v^{(1)} = 0 \quad \text{on} \quad z = x. \tag{3.7}$$

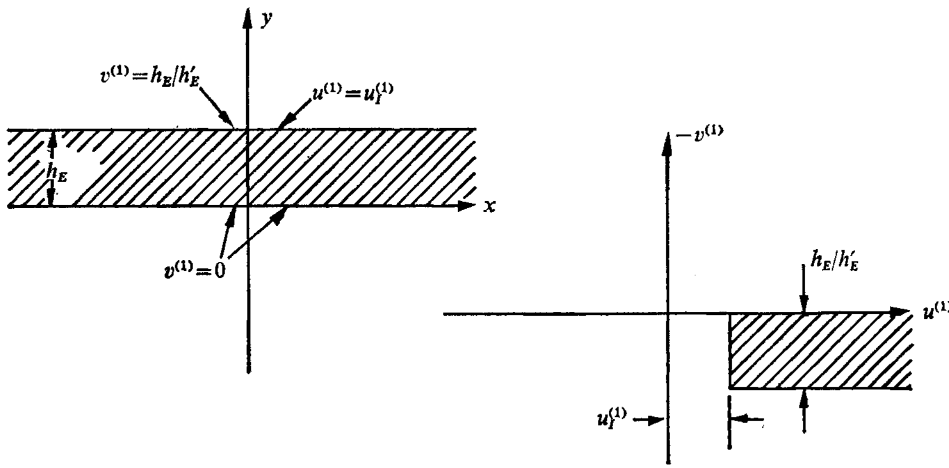

FIGURE 2. Flow boundaries and boundary conditions for first-order inner problem in physical and hodograph planes.

The constant $u_I^{(1)}$ must be found by matching with the outer solution for $\bar{x} < 0$. The problem for $q^{(1)}$, as defined above, is solved by noting that $q^{(1)}$ is the mapping of the strip $\{z: -\infty < x < \infty, 0 < y < h_E\}$ into the region of the $q^{(1)}$ plane indicated in figure 2. This gives the result that

$$q^{(1)} = u_I^{(1)} - ih'_E/h_E + [2h'_E/(\pi h_E)] \cosh^{-1} [i \exp(-\pi z/2h_E)]. \tag{3.8}$$

If we attempt to find $u_I^{(1)}$ by letting $x \to -\infty$ in (3.8), we find that $|q^{(1)}| = O(|x|)$, i.e. the solution is not uniformly valid. Instead, we use the matching rule, advocated by Van Dyke (1964, p. 90), that (in the notation of Fraenkel 1969)

$$(E_1 \bar{E}_1 - \bar{E}_1 E_1) q = 0 \tag{3.9}$$

with $z = x$ and $x < 0$. This gives in a straightforward manner

$$u_I^{(1)} = -\frac{2 \log 2}{\pi} \left(\frac{h'_E}{h_E}\right). \tag{3.10}$$

It turns out that in this problem the inner expansion is uniformly valid for all $\bar{x} > 0$, hence the outer expansion $\bar{q}$ is not needed for $\bar{x} > 0$. Even so, we match for $\bar{x} > 0$ to obtain

$$g_I^{(0)} = h_E \tag{3.11}$$

and

$$g_I^{(1)} = (2 \log 2/\pi) h_E h'_E. \tag{3.12}$$

The latter result permits us to find the contraction ratio C to $O(\epsilon)$, thus

$$C = \frac{h_E + \epsilon g_I^{(1)} + \dots}{h_E} = 1 + \epsilon h'_E \frac{2 \log 2}{\pi} + O(\epsilon^2). \tag{3.13}$$

The inner complex velocity can be decomposed into real and imaginary parts to get expressions for $u^{(1)}$ and $v^{(1)}$. Evaluating $v^{(1)}$ for $x > 0$ on $y = h_E$ and using condition (3.6a) shows that

$$g^{(1)} = \frac{4h'_E h_E}{\pi^2} \int_{\exp(-\pi x/2h_E)}^{1} \left[\frac{\sin^{-1} t}{t}\right] dt. \tag{3.14}$$

This can be integrated in terms of the dilogarithm function (Abramowitz & Stegun 1965, p. 1004), however, for the present purpose, asymptotic expressions for large and small x are more useful thus we find that

$$g^{(1)} = [(2/\pi)\log 2]\,h'_E h_E + O(e^{-\pi x/h_E}) \quad \text{as} \quad x \to \infty \tag{3.15}$$

and
$$g^{(1)} = h'_E x - \frac{4}{3\sqrt{\pi}}\frac{h'_E}{\sqrt{h_E}}\,x^{\frac{3}{2}} + O(x^{\frac{5}{2}}) \quad \text{as} \quad x \to 0. \tag{3.16}$$

The result for $x \to \infty$ shows that (3.14) is consistent with (2.7); the result for $x \to 0$ shows that the free surface is tangential to the nozzle edge.

4. Composite expansions

One strong advantage of using the matched asymptotic expansion formalism is the ease with which composite expansions can be constructed. To carry out this procedure for the present example, we use the additive rule of Van Dyke (1964, p. 95), i.e.

$$q_c = E_i q + \bar{E}_1 q - E_1 \bar{E}_1 q. \tag{4.1}$$

In constructing a composite expansion by this rule we are forced to take account of the fact that there are three regions where expansions of Poincaré form apply. These are the outer region with $\bar{x} < 0$, the inner region $-\lambda\epsilon < x < \infty$ and the outer region $\lambda\epsilon < \bar{x}$ (where λ is an arbitrary positive constant). It is true that the inner expansion is uniformly valid for the outer region with $\bar{x} > \lambda\epsilon$, however, when we use the additive composition rule on the inner expansion and the outer expansion for $\bar{x} < -\lambda\epsilon$, the resulting expansion will not be valid in the outer region for $\bar{x} > \lambda\epsilon$. This can also be seen by noting that $\bar{E}_1 E_1 q = E_1 \bar{E}_1 q$ depends on the sign of x (where we use $\bar{h}$ to order ϵ in the expression for $\bar{q}$). Therefore, the additive rule will give us two composite expansions, one valid for $x < \lambda\epsilon$ and the other (which is simply the inner expansion) valid for $x > -\lambda\epsilon$. Thus for $x < 0$ we have the composite expansion

$$q_c(x < 0) = \frac{1}{\bar{h}(\bar{z})} + \frac{2h'_E}{\pi h_E}\,\epsilon\left\{\log\left[\frac{\exp(-\pi z/2h_E) + [1+\exp(-\pi z/h_E)]^{\frac{1}{2}}}{2}\right] + \frac{\pi}{2h_E}z\right\}, \tag{4.2}$$

and for $x > -\lambda\epsilon$ we have

$$q_c(x > -\lambda\epsilon) = \frac{1}{h_E} + \frac{2h'_E}{\pi h_E}\,\epsilon\log\left[\frac{\exp(-\pi z/2h_E) + [1+\exp(-\pi z/h_E)]^{\frac{1}{2}}}{2}\right]. \tag{4.3}$$

The improvement gained by the composite expansion is that it is valid in the nozzle and the nozzle exit region; the second composite expansion (for the reasons stated above) is the inner solution. We have not been able to find a composite expansion for q that is uniformly valid in all three domains and which maintains the required analyticity requirement.

Most problems solved by the method of matched asymptotic expansions are too complicated to permit the consistency check (suggested by Fraenkel) of inserting the composite expansion into the equations and boundary conditions and verifying that all conditions are satisfied uniformly to a given order.

The present problem is one in which conditions are simple enough to carry out this calculation. The condition that q_c be a harmonic function is identically satisfied, as are the conditions that $v_c = 0$ on $y = 0$, $u_c \to 1$ as $x \to -\infty$, $u_c \to$ constant as $x \to +\infty$ and $|q_c| = O(1)$ as $z \to ih_E$. The conditions (2.1) and (2.2) still remain to be checked, and after a simple though tedious calculation we find

$$|v_c/u_c - \epsilon h'(\bar{x})| = O\,[\epsilon^2 h'(h'' - h'^2/h)] \quad \text{on} \quad y = h(\bar{x}), \quad \bar{x} < 0,$$

$$\left|\frac{v_c}{u_c} - \epsilon\frac{dg^{(1)}}{dx}\right| = O\left[\frac{2\epsilon^2 \ln 2}{\pi} h'_E \left(1 + \frac{\pi g^{(1)}(x)\, e^{-\pi x/h_E}}{h_E^2 \ln 2}\right)\right] \quad \text{on} \quad y = h_E + \epsilon g^{(1)}(x)$$

and

$$\left| |q_c|^2 - \left(\frac{1}{h_E} + \epsilon u_1^{(1)}\right)^2 \right| = O\left[\epsilon^2 \left(\frac{4h'_E}{h_E^{\frac{3}{2}}} g^{(1)}(x)\, e^{-\pi x/2h_E}\,(1 - e^{-\pi x/h_E})^{\frac{1}{2}} + v^{(1)2}(x, h_E)\right)\right]$$
$$\text{on} \quad y = h_E + \epsilon g^{(1)}(x), \quad x > 0.$$

The terms multiplying ϵ^2 in the order estimates are bounded for all x in the appropriate regions, establishing the consistency of the procedure.

5. Discussion

Another check on the asymptotic solution can be obtained by comparison with available exact and numerical solutions. A well-known exact result (Gilbarg 1960) exists for the jet flow from a nozzle formed by two inclined planes. For this case the exact contraction ratio is given by

$$C^{-1} = 2 - \pi^{-1} \sin\,[\tfrac{1}{2}\alpha\pi\,(f(\tfrac{1}{2} + \tfrac{1}{4}\alpha) - f(\tfrac{1}{4}\alpha) - 2/\alpha)], \tag{5.1}$$

where f is the dilogarithm and $\alpha\pi$ is the angle between the plates. This result can be expanded for small α to give

$$C = 1 - \alpha \log 2 + O(\alpha^2).$$

If one notes that our $\epsilon = \tan(\frac{1}{2}\alpha\pi)$ and $h'_E = -1$, it can be seen that to $O(\epsilon)$ our result agrees with the exact expression.

It would be of considerable interest to compare our results with solutions for jets from symmetric curved nozzles. Unfortunately, such solutions, which also satisfy the small ϵ assumption, are difficult to find. Larock (1969) obtained quasi-numerical solutions but his wall slopes in our inner region are highly variable. Thus, for a nozzle with a 45° lip he obtains $C = 0{\cdot}7242$, while our formula gives $C = 0{\cdot}55$.

N. S. Clarke (private communication) has derived a general expression for the contraction ratio of a nozzle which he can evalulate asymptotically under several circumstances, one of which is a slowly varying sinusoidal nozzle. In this case our results agree. He is also able to evaluate profiles with several sharp discontinuities, which lead to different results. The present paper treated a case with only one region where the 'slowly varying' assumption that motivated the outer or hydraulic expansion broke down, that being the exit region. Clarke has shown that when other places where the assumptions break down occur, i.e. when there are nozzle shape variations at the exit on the scale of the nozzle width, one obtains different values of the contraction ratio. One expects that these difficulties could be treated by the introduction of appropriate new inner regions to handle the breakdown of the outer expansion at these points. The main point of the above was to demonstrate how at least one region (the exit) could be treated by introduction of inner variables and use of a matching principle.

In conclusion, we see that matched expansion techniques are applicable in a reasonably simple way to classical jet flow problems. The ultimate utility of the method still remains to be proved, e.g. can the technique be of assistance in understanding time-dependent or axisymmetric flows?

The author would like to thank Dr Inge Ryhming of Institut CERAC for his valuable assistance, Institut CERAC for support of the work, a J.F.M. reviewer for his careful reading and suggested revisions of the first version of this paper and Dr N. S. Clarke for his interest and valuable comments pertaining to the contraction ratio.

REFERENCES

ABRAMOWITZ, M. & STEGUN, A. 1965 *Handbook of Mathematical Functions*. Dover.

FRAENKEL, L. E. 1969 On the method of matched asymptotic expansions. *Proc. Camb. Phil. Soc.* **65**, 209.

GILBARG, D. 1960 Jets and cavities. In *Handbuch der Physik*, vol. 9 (edited C. Truesdell), p. 341.

LAROCK, B. E. 1969 Jets from two-dimensional symmetric nozzles of arbitrary shape. *J. Fluid Mech.* **37**, 479.

VAN DYKE, M. 1964 *Perturbation Methods in Fluid Mechanics*. Academic.